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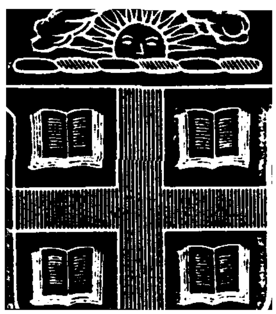

FUNCTIONAL OCCUPATION MEASURES AND ERGODIC COST PROBLEMS FOR SINGULARLY PERTURBED STOCHASTIC SYSTEMS

by

Harold J. Kushner

April 1989 #89-6

FUNCTIONAL OCCUPATION MEASURES AND ERGODIC COST PROBLEMS FOR SINGULARLY PERTURBED STOCHASTIC SYSTEMS *

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ABSTRACT.

Functional occupation measures are an extension to occupation measures on the path space of the usual definition of occupation measures for stochastic processes. They are used to get limit and approximation theorems for average cost per unit time problems for many types of controlled or uncontrolled random processes. In this paper, we deal with diffusions, reflected diffusions, and singularly perturbed controlled diffusions. There are extensions to wide bandwidth noise driven systems and to many other models. The method provides a convenient and powerful way of characterizing the processes associated with the weak limits of the occupation measures and with the sample limits of the average costs per unit time, as the various parameters of the problem go to their limits.The method can be used to get approximate optimality theorems and similar results for processes which are only approximated by jump diffusions and are of interest over a long time period.

INTRODUCTION.

In this paper, we develop a powerful tool for dealing with limit problems and approximations for controlled or uncontrolled processes which are of interest *over a long period of time*. The basic applications of interest concern ergodic cost problems for processes governed by either singularly perturbed stochastic differential equations or wide band noise driven systems (whether singularly perturbed or not). We want to show that certain "averaged" or "limit" systems can be used to get good approximations to the optimal value functions and controls for the "physical" systems as well as to characterize the "averaged" systems. Due to space limitations we concentrate on the singularly perturbed diffusion model and on reflected diffusions. It will be apparant that the general techniques of averaging and of the use of the "functional occupation measures" which are employed in this paper are readily usable for a wide variety of problems.

The fundamental mathematical technique involves a way of characterizing processes associated with the limits of certain occupation measures in

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a way that is mathematically interesting and readily usable. The occupation measures which have been used to date in weak convergence analyses are essentially occupation measures over the state space of the process. They measure the relative amount of time that the sample paths spend in the sets on the state space. The ***functional occupation measures*** are occupation measures over ***the sets in the path space***. Their main use here is in the characterization of the processes which yield the pathwise limits of "ergodic averages". They have been used in the large deviations literature [12], but in very different ways and with very different purposes. In this paper, we treat the background ideas, and illustrate some of the results which can be obtained.

For each ϵ, let $x^\epsilon(\cdot)$ denote a process with values in R^r, Euclidean r-space, and let (loosely speaking) $u^\epsilon(\cdot)$ be a control process for $x^\epsilon(\cdot)$. For bounded and continuous $k(\cdot)$, define $\gamma_T^\epsilon = \int_0^T k(x^\epsilon(s), u^\epsilon(s))ds/T$. The $x^\epsilon(\cdot)$ might be, for example, a singularly perturbed diffusion or a wide band noise driven system. In the singular perturbation case, the ϵ indexes the singular perturbation. In the wide band noise case, the ϵ indexes the inverse of the bandwidth. We are interested in the limits γ of the γ_T^ϵ as $\epsilon \to 0$ and as $T \to \infty$, and in the approximation of $x^\epsilon(\cdot)$ by "simpler " processes $x(\cdot)$ which are useful for approximating the values of functionals of $x^\epsilon(\cdot)$ for small ϵ and large T. These simpler processes are stationary and have mean average cost per unit time γ. In the controlled case, we want controls that are "good" or nearly optimal for $x(\cdot)$ to be "good" or nearly optimal for the $x^\epsilon(\cdot)$ for small ϵ and large T. The functional occupation measure method facilitates the characterization of these simpler processes. It uses weak *convergence* techniques connected with the sequence of functional occupation measures for the $x^\epsilon(\cdot)$.

In order to introduce the ideas in a simple way, we start with the classical occupation measures and use the following assumptions

A1.1 $b(\cdot), \sigma(\cdot)$ are continuous functions with a growth at most linear as $|x| \to \infty$.

Let $w(\cdot)$ denote a standard vector-valued Wiener process. Define the process $x(\cdot)$ as the solution to

$$dx = b(x) + \sigma(x)dw, x \in R^r. \tag{1.1}$$

Let A denote the differential generator of $x(\cdot)$.

A1.2. $\{x(t), t < \infty\}$ is bounded in probability.

A1.3. (1.1) has a unique invariant measure $\mu(\cdot)$.

If the state space of $x(\cdot)$ is not bounded, then conditions such as (A1.2) are usually verified via a stochastic Liapunov function method. Let $\mathcal{B}(S)$ denote the Borel sets of the metric space S. Define the occupation measure $Q^t(\cdot)$ by

$$Q^t(B) = I_{\{x(t)\}}(B), \quad B \in (R^r),$$

and the normalized occupation measure

$$Q_T(B) = \frac{1}{T}\int_0^T Q^t(B)dt.$$

Let $C(S)$ denote the continuous real valued functions on S, $C_b(S)$ the subset of bounded functions, $C_0(S)$ the subset of functions with compact support, and $C_0^2(R^r)$ the further subset of functions whose mixed partial derivatives are continuous-up to second order, when $S = R^r$.

It is well known that [1] under appropriate conditions, the sequence of measure-valued random variables $\{Q_T(\cdot)\}$ converges in probability (in the weak topology) to the invariant measure $\mu(\cdot)$, as $T \to \infty$; i.e., for each $f(\cdot) \in C_b(R^r)$, $\int f(x)Q_T(dx) \to \int f(x)\mu(dx)$ in probability as $T \to \infty$. Various extensions of this result to problems in stochastic control theory were used in [2] to prove the existence of an optimal feedback control for an ergodic cost problem for a controlled version of (1.1). They were used in [3] and [4] to get many approximation and convergence results for ergodic cost problems for wide band noise driven systems operating over a very long time interval.

In this paper, we take a point of view which is more general and which which has numerous practical advantages. Let $D^r[0,\infty)$ denote the space of R^r-valued functions which are right continuous and have left hand limits and which is endowed with the Skorohod topology [5, Chapter 3.5]. The space $D^r[0,\infty)$ is complete and separable, and we let $\phi(\cdot)$ denote its generic element. For $\phi(\cdot)$ in $D^r[0,\infty)$, define the shifted function $\phi_t(\cdot) = \phi(t + \cdot)$. Define the shifted process $x_t(\cdot) = x(t + \cdot)$, and define the occupation measure $\tilde{P}^t(\cdot)$ by $\tilde{P}^t(B_0) = I_{\{x_t(\cdot)\}}(B_0)$ and the *functional occupation measure*

$$\tilde{P}_T(B_0) = \frac{1}{T}\int_0^T \tilde{P}^t(B_0)dt. \tag{1.2}$$

Under (A1.2), $\{\tilde{P}_T(\cdot), T < \infty\}$ is a tight sequence of measure-valued random variables (see Sections 2 and 3 for the exact definitions and proofs). Let $\tilde{P}_T^\omega(\cdot)$ denote the sample value of $\tilde{P}_T(\cdot)$. For almost all ω, $\tilde{P}_T^\omega(\cdot)$ is a measure on $\mathcal{B}(D^r[0,\infty))$, the Borel sets of $D^r[0,\infty)$, and hence induces a process with paths in $D^r[0,\infty)$. Let $\tilde{P}(\cdot)$ denote a measure-valued random variable which is a weak limit of a weakly convergent subsequence of $\{\tilde{P}_T(\cdot), T < \infty\}$. Suppose that the $\tilde{P}_T(\cdot)$ and the $\tilde{P}(\cdot)$ are defined on the

same sample space with generic variable ω. Let $\tilde{P}^{\omega}(\cdot)$ denote the sample value of $\tilde{P}(\cdot)$. Then, for almost all ω, the measure $\tilde{P}^{\omega}(\cdot)$ also induces a process on $D^r[0,\infty)$. It turns out (Theorem 3.2) that this process, which we write as $x^{\omega}(\cdot)$, is a stationary diffusion of the form (1.1). Since ω indexes the *sample value of the measure* $\tilde{P}(\cdot)$, it indexes the entire process $x^{\omega}(\cdot)$. and *not* the sample values of that process. The necessary probability background appears in Section 2. The proofs are developed in Section 3. We spend a lot of time on the simple case (1.1), since it allows a relatively unencumbered treatment of the basic ideas of the functional occupation measure method. Section 4 gives the details of an application to a control problem for a controlled form of (1.1). Applications to the singularly perturbed control problem on the infinite time interval are dealt with in Section 5. Here the $\tilde{P}^{\epsilon}(\cdot)$ are for a singularly perturbed controlled diffusion, and we show that the limit of the samples of the $\tilde{P}_T(\cdot)$ induce stationary controlled diffusions for an "averaged system". Then various approximate optimality theorems are proved. An extension of the results of Section 3 to a reflected diffusion appears in Section 6. The aim is to outline some of the possibilities. Further applications will appear in [6].

2. PROBABILITY PRELIMINARIES.

Definitions. Let S denote a metric space with metric $d(\cdot)$, and for each set $A \in \mathcal{B}(S)$, define the set $A^{\epsilon} = \{x : d(x,y) \leq \epsilon \text{ for some } y \in A\}$. Let $\mathcal{P}(S)$ denote the collection of probability measures on $(S, \mathcal{B}(S))$. The Prohorov metric [5,p 96] $\pi(\cdot)$ on $\mathcal{P}(S)$ is defined by

$$\pi(P,P') = \inf\{\epsilon > 0 : P'(A) \leq P(A^{\epsilon}) + \epsilon, \text{ for all compact } A \in \mathcal{B}(S)\}$$

A set $\{P_{\alpha}(\cdot)\} \in \mathcal{P}(S)$ is said to be *tight* if for each $\delta > 0$, there is a compact set $K_{\delta} \in \mathcal{B}(S)$ such that $\sup_{\alpha} P_{\alpha}\{x \notin K_{\delta}\} \leq \delta$. We say that $\{P_n(\cdot)\}$ in $\mathcal{P}(S)$ *converges weakly* to $P(\cdot)$ in $\mathcal{P}(S)$ and is written $P_n \Rightarrow P$ iff $(P_n, f) \equiv \int f(x) P_n(dx) \to (P, f)$ for all $f(\cdot) \in C_b(S)$. We have

Theorem 2.1. [4,p101]. *If $(S, d(\cdot))$ is complete and separable then so is the metric space $(\mathcal{P}(S), \pi(\cdot))$.*

Theorem 2.2. [4, p104]. (Prohorov's Theorem) *Let S be complete and separable. Then a set $M \subset \mathcal{P}(S)$ is relatively compact iff M is tight. Also, $P_n \Rightarrow P$ is equivalent to $\pi(P_n, P) \to 0$. If S is compact, then the topology of weak convergence is equivalent to the topology under the Prohorov metric.*

Random variables. Let X_n and X be S-valued random variables with associated measures $P_n(\cdot)$ and $P(\cdot)$, resp. We use the terminology *tightness* of $\{X_{\alpha}\}$ and *weak convergence* of X_n to X (written $X_n \Rightarrow X$) interchangably with the terminlogy *tightness* of $\{P_{\alpha}(\cdot)\}$ and the *weak convergence* $P_n \Rightarrow P$. The sense of the usage of the words *tightness* and *weak*

convergence depends on whether the objects in question are measures or random variables, and the abuse of notation should cause no problems. The following "Skorohod representation" will be very useful.

Theorem 2.3. [4,p102]. *Let S be separable, and let $P_n(\cdot)$ and $P(\cdot)$ be the measures of the S-valued random variables X_n and X, resp. Let $X_n \Rightarrow X$. Then there exists a probability space $(\hat{\Omega}, \hat{P}, \hat{\mathcal{F}})$ with S-valued random variables $\{\hat{X}_n\}$ and $\hat{X}$ defined on it, such that $\hat{X}_n$ and $\hat{X}$ have the distributions of X_n and X, resp., and $d(\hat{X}_n, \hat{X}) \to 0$ w.p.1.*

We will use the Skorohod representation quite frequently since it simplifies the calculations by allowing us to assume that the weak convergence is actually w.p.1 convergence in the appropriate topology. In order to simplify the notation, when using the Skorohod representation we will not use the "hat" symbol for the new sequence, but will retain whatever the original notation was for the sequence.

Measure-valued random variables and processes. Let S_0 be a complete and separable metric space. Then $\mathcal{P}(S_0) = S_1$ is a complete and separable metric space. Let Q_n and Q be S_1-valued random variables. Their probability distributions are, of course, elements of $\mathcal{P}(S_1)$. Suppose that $\{Q_n\}$ is tight (in the sense of a sequence of random variables) and let $Q_n \Rightarrow Q$. If Q is a measure-valued random variable, then we write the sample value as Q^ω or as $Q^\omega(\cdot)$. Suppose now that the Skorohod representation is used so we can assume that $\{Q_n(\cdot), n < \infty, Q(\cdot)\}$ are defined on the same sample space and that the generic variable of the space is ω. By the weak convergence and the Skorohod imbedding $Q_n^\omega(\cdot) \to Q^\omega(\cdot)$ for almost all ω in the topology of S_1. I.e.,

$$\pi(Q_n^\omega, Q^\omega) \to 0 \tag{2.1}$$

for almost all ω. For each ω, the value $Q_n^\omega(\cdot)$ is a measure on S_0. Let ω_0 denote the canonical point of S_0. Then, for each ω, $Q_n^\omega(\cdot)$ induces an S_0-valued random variable which we denote by X_n^ω and whose values are denoted by $X_n^\omega(\omega_0)$. Note that the ω and n index the *measure*. With the measure $Q_n^\omega(\cdot)$ given, ω_0 gives the sample value of the *random variable associated with that measure*.

Now, Theorem 2.2 and (2.1) imply that for almost all ω

$$X_n^\omega \Rightarrow X^\omega. \tag{2.2}$$

By (2.2), we can use the Skorohod representation for the set of random variables $(\{X_n^\omega\}, X^\omega)$ for each fixed ω not in some null set and suppose that they are defined on some sample space with generic variable ω_0. Then for almost all ω_0,

$$d(X_n^\omega(\omega_0), X^\omega(\omega_0)) \to 0. \tag{2.3}$$

For our purposes it will usually be sufficient to work with each ω, and then the relation (2.3) will be quite useful. The following theorem provides a

useful way of proving that a sequence of $\mathcal{P}(S_0)$-valued random variables $\{Q_\alpha\}$ is tight.

Theorem 2.4. *Suppose that there are compact $S_n \uparrow S_0$ such that*

$$\lim_n \sup_\alpha P\{Q_\alpha(S_0 - S_n) > \frac{1}{n}\} = 0 \tag{2.5}$$

or, equivalently,

$$\lim_n \sup_\alpha EQ_\alpha(S_0 - S_n) = 0. \tag{2.6}$$

Then $\{Q_\alpha\}$ is tight (as a sequence of random variables).

Proof. Given $\delta > 0$, we need to find a compact set K_δ in $\mathcal{P}(S_0)$ such that $\sup_\alpha P\{Q_\alpha \notin K_\delta\} \leq \delta$. By (2.5) for each $\epsilon > 0$ there is a compact set S_ϵ such that

$$\sup_\alpha P\{Q_\alpha(S_0 - S_\epsilon) \geq \epsilon\} \leq \epsilon.$$

Let $\epsilon_i \to 0$ and $\sum_i \epsilon_i < \infty$. Given $\delta > 0$, choose m_δ such that $\sum_{i=m_\delta}^{\infty} \epsilon_i \leq \delta$. Define the precompact set

$$K_\delta = \{Q \in \mathcal{P}(S_0) : Q(S_0 - S_{\epsilon_i}) \leq \epsilon_i, i \geq m_\delta\}.$$

Since $\sup_\alpha P\{Q_\alpha \notin K_\delta\} \leq \delta$, the proof is completed. Q.E.D.

3. LIMITS OF OCCUPATION MEASURES FOR DIFFUSIONS

We return to the special case of Section 1, where the system is (1.1), and $\tilde{P}_T(\cdot)$ is defined by (1.2). Define $S_0' = D^r[0,\infty)$. Theorem 3.1 gives a criterion for the tightness of the normalized occupation measures.

Theorem 3.1. *Assume (A1.1) and (A1.2). Then $\tilde{P}_T(\cdot)$ is a $\mathcal{P}(S_0')$-valued random variable, and $\{\tilde{P}_T(\cdot), T < \infty\}$ is tight.*

Proof. For each $B \in \mathcal{B}(S_0)$ the measurability of the process $x(\cdot)$ implies the (ω, t)-measurability of the function with values $\tilde{P}^t(\omega, B)$. Thus $\tilde{P}_T(\cdot)$ is a random variable for each T. By (A1.1) and (A1.2), the sequence of processes $\{x_t(\cdot)\}$ is tight. Thus for each $\delta > 0$, there is a compact set K_δ in S_0' such that $P\{x_t(\cdot) \in S_0' - K_\delta\} \leq \delta$, all t. This implies that

$$E\tilde{P}_T\{S_0' - K_\delta\} = \frac{1}{T}\int_0^T E\tilde{P}^t\{S_0' - K_\delta\}dt \leq \delta,$$

and the tightness of $\{\tilde{P}_T(\cdot), T < \infty\}$ follows from Theorem 2.4. Q.E.D.

A remark on notation. Let $\{\tilde{P}_{T_n}(\cdot), n < \infty\}$ be a weakly convergent subsequence with limit denoted by $\tilde{P}(\cdot)$. Let ω be the generic variable on

the sample space on which $x(\cdot)$ is defined. Then on this sample space, for each ω, we have the representation

$$\tilde{P}_T^{\omega}(G) = \frac{1}{T}\int_0^T I_{\{x_t^{\omega}(\cdot)\}}(G)dt, \tag{3.1a}$$

$$\tilde{P}_T(G) = \frac{1}{T}\int_0^T I_{\{x_t(\cdot)\}}(G)dt. \tag{3.1b}$$

Now, let us use the Skorohod representation for $\{\tilde{P}_{T_n}, n < \infty, \tilde{P}(\cdot)\}$, and let the generic variable of the common sample space be ω. Then, using the same notation $\{\tilde{P}_{T_n}(\cdot), n < \infty, \tilde{P}(\cdot)\}$for the imbedded random variables, (3.1b) continues to hold in that the distributions of the two sides are equal for each T and G. Let $=^D$ denote equality in the sense of distribution.

Theorem 3.2 *Assume (A1.1), (A1.2) and let $T_n \to \infty$ index a weakly convergent subsequence of $\{\tilde{P}_T(\cdot), T < \infty\}$ with limit denoted by $\tilde{P}(\cdot)$.*

Assume that the Skorohod representation is used and that ω denotes the generic variable of the sample space on which $\{\tilde{P}_{T_n}(\cdot), n < \infty, \tilde{P}(\cdot)\}$ are defined. Then, for almost all ω, the sample value $\tilde{P}^{\omega}(\cdot)$ of $\tilde{P}(\cdot)$ induces a stationary diffusion process $\tilde{x}^{\omega}(\cdot)$ on S_0' with differential operator A. Let $\tilde{\mu}^{\omega}(\cdot)$ denote the stationary measure of $\tilde{x}^{\omega}(\cdot)$. Let $K(\cdot)$ denote a bounded and continuous real valued function on S_0'. Then

$$\frac{1}{T_n}\int_0^{T_n} K(x_t(\cdot))dt \Rightarrow \int K(\phi(\cdot))\tilde{P}(d\phi)$$
$$= EK(\tilde{x}^{\omega}(\cdot)). \tag{3.2}$$

For $k(\cdot) \in C_b(R^r)$

$$\frac{1}{T_n}\int_0^{T_n} k(x(s))ds \Rightarrow \int k(x)\tilde{\mu}(dx), \tag{3.3}$$

where $\tilde{\mu}(\cdot)$ is the measure valued (on the Borel sets of R^r) random variable with values $\tilde{\mu}^{\omega}(\cdot)$ Under (A1.3), for almost all ω

$$\tilde{\mu}^{\omega}(\cdot) = \mu(\cdot).$$

and the subsequence is irrelevent.

Proof. *Stationarity.* We use the Skorohod representation for $\{\tilde{P}_{T_n}(\cdot), n < \infty, \tilde{P}(\cdot)\}$ all through the proof. Let $G \in S_0'$, and define its left shift $G_c = \{\phi(\cdot) : \phi_c(\cdot) \in G\}$. Then

$$\tilde{P}_T(G) =^D \frac{1}{T}\int_0^T I_{\{x_t(\cdot)\}}(G_c)dt = \frac{1}{T}\int_0^T I_{\{x_{t-c}(\cdot)\}}(G)dt$$

$$\tilde{P}_T(G_c) - \tilde{P}_T(G) =^D \frac{1}{T}\int_T^{T+c} I_{\{x_t(\cdot)\}}(G)dt - \frac{1}{T}\int_0^c I_{\{x_t(\cdot)\}}(G)dt.$$

Thus $\tilde{P}_T^\omega(G) - \tilde{P}_T^\omega(G_c) \to 0$ as $T \to 0$ for all ω and G. This and the weak convergence implies that $\tilde{P}^\omega(G) = \tilde{P}^\omega(G_c)$ for all ω and all sets G with $\tilde{P}^\omega(\partial G) = 0$. This, in turn, implies the stationarity.

In the rest of the proof, we will only characterize the process induced by the values $\tilde{P}^\omega(\cdot)$ of the limit $\tilde{P}(\cdot)$. By the definition of the sample occupancy measure $\tilde{P}_T^\omega(\cdot)$, for each real-valued, bounded and measurable function $F_0(\cdot)$ on S_0', we have

$$\int F_0(\phi)\tilde{P}_T(d\phi) =^D \frac{1}{T}\int_0^T F_0(x_t(\cdot))dt \tag{3.4}$$

In fact, (3.4) completely determines $\tilde{P}_T(\cdot)$.

Let $h(\ )$ be a bounded and continuous real valued function of its' arguments, let q be an arbitrary integer, and let $t_i, \tau, \tau+s$ be such that $t_i \le \tau \le \tau+s, i \le q$. Let $f(\cdot) \in C_0^2(R^r)$. Define the function $F(\cdot)$ by

$$F(\phi(\cdot)) = h(\phi(t_i), i \le q)[f(\phi(\tau+s)) - f(\phi(\tau)) - \int_\tau^{\tau+s} Af(\phi(u))du]. \tag{3.5}$$

The function $F(\cdot)$ is measurable but it is not continuous at all points $\phi(\cdot)$. It is continuous at each $\phi(\cdot)$ which is continuous [6, p121].

Since the processes induced on S_0' by the $\tilde{P}_T^\omega(\cdot)$ are continuous for each ω and T, the sample values $\tilde{P}^\omega(\cdot)$ of the limit measures also have their support on the set of continuous functions in S_0'. Hence, even though the function $F(\cdot)$ is not continuous at every point in S_0', it is continuous w.p.1 relative to $\tilde{P}^\omega(\cdot)$ for almost all ω, since it is continuous at each point $\phi(\cdot)$ which is a continuous function. Due to this fact and the Skorohod representation [7, p5.1],

$$\int F(\phi)\tilde{P}_{T_n}^\omega(d\phi) \to \int F(\phi)\tilde{P}^\omega(d\phi). \tag{3.6}$$

for almost all ω. Let $\tilde{E}^\omega$ and $\tilde{x}^\omega(\cdot)$ denote the expectation with respect to $\tilde{P}^\omega(\cdot)$ and the process which is induced by $\tilde{P}^\omega(\cdot)$, resp. The result (3.2) is just a consequence of the weak convergence. By the definition of $F(\cdot)$, the right hand side of (3.4), evaluated at ω equals

$$\int F(\phi)\tilde{P}^\omega(d\phi) = \tilde{E}^\omega h(\tilde{x}^\omega(t_i), i \le q)\cdot$$

$$[f(\tilde{x}^\omega(\tau+s)) - f(\tilde{x}^\omega(\tau)) - \int_\tau^{\tau+s} Af(\tilde{x}^\omega(u))du] \tag{3.7}$$

It will be shown below that (3.7) equals zero for almost all ω. Thus, owing to the arbitrariness of $h(\cdot), \tau, \tau+s, q, t_i, f(\cdot)$, for almost all ω the process $\tilde{x}^{\omega}(\cdot)$ must solve the martingale problem for the operator A.

Let $K(\phi(\cdot)) = k(\phi(0))$. Then, for almost all ω, $K(\cdot)$ is continuous w.p.1 relative to the measures $\tilde{P}^{\omega}(\cdot)$. Note that

$$\frac{1}{T}\int_0^T k(x(s))ds = \frac{1}{T}\int_0^T K(x_s(\cdot))ds =^D \int K(\phi)\tilde{P}_T(d\phi),$$

$$\int K(\phi)\tilde{P}_T^{\omega}(d\phi) \to \int K(\phi)\tilde{P}^{\omega}(d\phi) = \int k(\phi(0))\tilde{P}^{\omega}(d\phi(0)).$$

But by the stationarity of $\tilde{x}^{\omega}(\cdot)$ for almost all ω, this last expression on the right equals (for almost all ω) $\int k(x)\tilde{\mu}^{\omega}(dx)$.

Proof that (3.7) equals zero for almost all ω. By Ito's Formula and the representation (3.4), we can write

$$\int F(\phi)\tilde{P}_T^{\omega}(d\phi) =^D$$

$$\frac{1}{T}\int_0^T \Big(h(x_t(t_i), i \le q)\Big[\int_{\tau}^{\tau+s} f_x'(x_t(u)\sigma(x_t(u))dw_t(u)\Big]\Big)dt$$

$$= \frac{1}{T}\int_0^T dt \int_{t+\tau}^{t+\tau+s} g(v)dw(v)$$

for some bounded and non-anticipative $g(\cdot)$. This expression goes to zero in probability as $T \to \infty$. This fact together with the weak convergence implies that (3.7) equals zero for almost all ω. Q.E.D.

4. THE CONTROL PROBLEM.

Definitions In order to present the results in a relatively simple way, we work with stochastic relaxed controls [3], [8]. Let U, the space of values for the control, be a compact set in some Euclidean space. Let $\mathcal{F}_t$ denote a filtration and $w(\cdot)$ a standard vector-valued $\mathcal{F}_t$-Wiener process. Let $\mathcal{M}_1(U \times [0,\infty)) \equiv S_0''$ denote the space of measures $\nu(\cdot)$ on $\mathcal{B}(U \times [0,\infty))$ with the following property: $\nu(U \times [0,t]) \equiv \nu(U,t) = t$, all t. Such $\nu(\cdot)$ are called (deterministic) *relaxed controls*. On S_0'' we put the "compact-weak" topology; i.e., $\nu_n(\cdot) \to \nu(\cdot)$ iff $(\phi,\nu_n) \to (\phi,\nu)$ for each $\phi(\cdot)$ in $C_0(U \times [0,\infty))$. Under this topology, S_0'' is a complete and separable metric space. If $\nu(\cdot) \in S_0''$, then there is a *derivative* $\nu_t(\cdot)$ such that $\nu_t(\cdot)$ is a measure on $\mathcal{B}(U)$ for each t, $\nu_t(C)$ is t-measurable for each $C \in \mathcal{B}(U)$, and $\nu(d\alpha dt) = \nu_t(d\alpha)dt$. Define $S_0 = S_0' \times S_0''$.

An *admissible stochastic relaxed control* $m(\cdot)$ is an S_0''-valued random variable such that $m(C,t)$ is $\mathcal{F}_t$-adapted and $m(C,\cdot)$ is measurable for each $C \in \mathcal{B}(U)$. The derivative $m_t(\cdot)$ can be constructed so that for each

$C \in \mathcal{B}(U)$. $m_t(C)$ is $\mathcal{F}_t$-adapted. If $w(\cdot)$ is an $\mathcal{F}_t-$ Wiener process, we sometimes say that $(m(\cdot), w(\cdot))$ is admissible or that $m(\cdot)$ is admissible with respect to $w(\cdot)$.

For admissible $(m(\cdot), w(\cdot))$ the controlled SDE is written as

$$dx = \int_U b(x, \alpha) m_t(d\alpha) dt + \sigma(x) dw, \tag{4.1}$$

We use the following replacement for (A1.1).

A4.1 *$\sigma(\cdot)$ and $b(\cdot)$ are continuous, $\sigma(\cdot)$ and $b(\cdot, \alpha)$ have at most a linear growth as $|x| \to \infty$, uniformly in α.*

Let A^α denote the differential generator of (4.1) with the control fixed at $\alpha \in U$. For an admissible stochastic relaxed control $m(\cdot)$, let A^m denote the differential generator of $x(\cdot)$. Then, acting on smooth functions of x at time t, we can write $A^m f(x)(t) = \int A^\alpha f(x) m_t(d\alpha)$. Define the "shifted" stochastic relaxed control $\Delta_t m(\cdot) = m(t + \cdot) - m(t)$. The relaxed control $\Delta_t m(\cdot)$ is the one that is actually used on the shifted process $x_t(\cdot)$, since $(\Delta_t m)_s(\cdot) = m_{t+s}(\cdot)$ and

$$dx_t(s) = ds \int_U b(x_t(s), \alpha) m_{t+s}(d\alpha) + \sigma(x_t(s)) dw_t(s).$$

Define the occupation measures $\tilde{P}^t(\cdot)$ and $\tilde{P}_T(\cdot)$ by

$$\tilde{P}^t(B_0 \times C_0) = I_{\{x_t(\cdot)\}}(B_0) I_{\{\Delta_t m(\cdot)\}}(C_0),$$

$$\tilde{P}_T(\cdot) = \frac{1}{T} \int_0^T \tilde{P}^t(\cdot) dt.$$

Here, $C_0 \in \mathcal{B}\{S_0''\}$ and $B_0 \in \mathcal{B}\{S_0'\}$.

Theorem 4.1. *Assume (A1.2) and (A4.1). Then $\{\tilde{P}_T(\cdot), T < \infty\}$ is a tight set of $\mathcal{P}(S_0)$-valued random variables. Let $\tilde{P}(\cdot)$ denote the limit of some weakly convergent subsequence (indexed by $T_n \to \infty$), and suppose that the Skorohod representation is used for $\{\tilde{P}_{T_n}(\cdot), n < \infty, \tilde{P}_T(\cdot)\}$. Then, for almost all ω, $\tilde{P}^\omega(\cdot)$ induces a process $(\tilde{x}^\omega(\cdot), \tilde{m}^\omega(\cdot))$ on S_0 and the distribution of the shifted pair $(\tilde{x}_t^\omega(\cdot), \Delta_t \tilde{m}^\omega(\cdot))$ does not depend on t. For each ω, there is a filtration $\tilde{\mathcal{F}}_t^\omega$ and an $\tilde{\mathcal{F}}_t^\omega$-standard vector-valued Wiener process $\tilde{w}^\omega(\cdot)$, such that $\tilde{m}^\omega(\cdot)$ is admissible (with respect to $\tilde{w}^\omega(\cdot)$) and*

$$d\tilde{x}^\omega = \int_U b(\tilde{x}^\omega, \alpha) \tilde{m}_t^\omega(d\alpha) dt + \sigma(\tilde{x}^\omega) d\tilde{w}^\omega.$$

Define the random variable K with values

$$\tilde{E}^\omega \int_0^1 \int_U k(\tilde{x}^\omega(s), \alpha) \tilde{m}_s^\omega(d\alpha) ds.$$

Then

$$\frac{1}{T_n}\int_0^{T_n}\int_U k(x(t),\alpha)m_t(d\alpha)dt \Rightarrow \bar{K} \tag{4.2}$$

Proof. The proof is similar to that of Theorem 3.2. The set of processes $\{\Delta_t m(\cdot), t < \infty\}$ is tight and so is the set $\{x_t(\cdot), t < \infty\}$. This implies the tightness of $\{\tilde{P}_T(\cdot), T < \infty\}$, and we need only characterize the limits of the weakly convergent subsequences and explain (4.2). Let T_n index a weakly convergent subsequence, with limit denoted by $\tilde{P}(\cdot)$. Let $(\tilde{x}^\omega(\cdot), \tilde{m}^\omega(\cdot))$ denote the process induced by $\tilde{P}^\omega(\cdot)$. The asserted stationarity is proved as in Theorem 3.2. Let $\psi(\cdot)$ and $\psi_j(\cdot)$ be in $C_0(U \times [0,\infty))$ and define $(\psi,\nu)_t = \int_U \int_0^t \psi(s,\alpha)\nu(d\alpha ds) = \int_U \int_0^t \psi(s,\alpha)\nu_s(d\alpha)ds$. Choose $h(\cdot), f(\cdot), q, t_i, \tau, \tau+s$ as in Theorem 3.2 and let p be an arbitrary integer. Define the function $F(\cdot)$ on S_0 by

$$F(\phi,\nu) = h(\phi(t_i),(\psi_j,\nu)_{t_i}, i \le q, j \le p)[f(\phi(\tau+s)) - f(\phi(\tau)) - \int_\tau^{\tau+s}\int_U A^\alpha f(\phi(u))\nu(d\alpha du)].$$

By Ito's Formula and the definition of $\tilde{P}_T(\cdot)$,

$$\int F(\phi,\nu)\tilde{P}_T(d\phi d\nu) \stackrel{D}{=} \frac{1}{T}\int_0^T F(x_t(\cdot),\Delta_t m(\cdot))dt = \tag{4.3}$$

$$\frac{1}{T}\int_0^T h(x_t(t_i),(\psi_j,\Delta_t m)_{t_i}, i \le q, j \le p)dt \int_\tau^{\tau+s} f_x'(x_t(u))\sigma(x_t(u))dw_t(u)$$

Let us use the Skorohod representation. By the same argument used in Theorem 3.2, we have, for almost all ω,

$$\int F(\phi,\nu)\tilde{P}(d\phi d\nu) = 0. \tag{4.4}$$

Equivalently, for almost all ω, and with $\tilde{E}^\omega$ denoting the expectation with respect to $\tilde{P}^\omega(\cdot)$,

$$\tilde{E}^\omega h(\tilde{x}^\omega(t_i),(\psi_j,\tilde{m}^\omega)_{t_i}, i \le q, j \le p)[f(\tilde{x}^\omega(\tau+s)) - f(\tilde{x}^\omega(\tau)) - \int_\tau^{\tau+s}\int_U A^\alpha f(\tilde{x}^\omega(u))\tilde{m}^\omega(d\alpha du)] = 0, \text{ w.p.1.}$$

Analogous to the situation in Theorem 3.2, the process $\tilde{x}^\omega(\cdot)$ solves the martingale problem with operator $A^{\tilde{m}^\omega}$ and with respect to the filtration $\mathcal{B}(\tilde{x}^\omega(s), \tilde{m}^\omega(\cdot,s), s \le t)$. All the conclusions except the representation on the right hand side of (4.2) follows from this.

Define the function $K(\cdot)$ on S_0 by

$$K(\phi,\nu) = \int_0^1 \int_U k(\phi(s),\alpha)\nu(dsd\alpha).$$

The right hand side of (4.2) is obtained by noting that the limit of the left hand side of (4.2) equals the limit of

$$\frac{1}{T_r}\int_0^{T_n} K(x_t(\cdot),\Delta_t m(\cdot))dt \stackrel{D}{=} \int K(\phi,\nu)\tilde{P}_T(d\phi d\nu).$$

Q.E.D.

5. SINGULARLY PERTURBED CONTROL PROBLEMS: LIMIT AND APPROXIMATION THEOREMS

In this section, we work with the singularly perturbed diffusion model

$$dx^\epsilon = [\int G(x^\epsilon, z^\epsilon, \alpha)m_t^\epsilon(d\alpha)]dt + \sigma(x^\epsilon, z^\epsilon)dw_1, \tag{5.1}$$

$$\epsilon dz^\epsilon = H(x^\epsilon, z^\epsilon)dt + \sqrt{\epsilon}v(x^\epsilon, z^\epsilon)dw_2, \tag{5.2}$$

$$a(x,z) = \sigma(x,z)\sigma'(x,z).$$

The $w_i(\cdot)$ are standard vector valued Wiener processes with respect to some filtration $\mathcal{F}_t$, and $m^\epsilon(\cdot)$ is an admissible relaxed control (always with respect to $\mathcal{F}_t$ or $(w_1(\cdot), w_2(\cdot))$). The control takes values in a compact set U. The model (5.1), (5.2), is the most common one used for stochastic control problems under singular perturbations [6],[9],[10]. The model will allow a relatively simple and generic development of the averaging methods.

Definition. We say that the solution to (5.1), (5.2) is *unique in the weak sense* if the distribution of $(m^\epsilon(\cdot), w_1(\cdot), w_2(\cdot))$ determines that of

$$(x^\epsilon(\cdot), z^\epsilon(\cdot), m^\epsilon(\cdot), w_1(\cdot), w_2(\cdot))$$

. We use the analogous definition in the absence of a control. We use $E_{x,z}^{m^\epsilon}$ to denote the expectation under the control $m^\epsilon(\cdot)$, and with the initial condition (x,z). The distribution of $(x^\epsilon(\cdot), z^\epsilon(\cdot))$ actually depends on the joint distribution of $(w_1(\cdot), w_2(\cdot), m^\epsilon(\cdot))$, but we omit the Wiener processes from some of the notation for simplicity.

The fixed-x, rescaled "fast process". We exploit the time scale differences between $x^\epsilon(\cdot)$ and $z^\epsilon(\cdot)$ in order to approximate (5.1) by a simpler "averaged" system. Define the "stretched out" processes $z_0^\epsilon(t) = z^\epsilon(\epsilon t)$, $x_0^\epsilon(t) = x^\epsilon(\epsilon t)$. Then

$$dx_0^\epsilon = [\epsilon \int G(x_0^\epsilon, z_0^\epsilon, \alpha)m_t^\epsilon(d\alpha)]dt + \sqrt{\epsilon}\sigma(x_0^\epsilon, z_0^\epsilon)d\tilde{w}_1 \tag{5.3}$$

$$dz_0^\epsilon = H(x_0^\epsilon, z_0^\epsilon)dt + v(x_0^\epsilon, z_0^\epsilon)d\tilde{w}_2. \tag{5.4}$$

where the $\tilde{w}_i(\cdot)$ are standard Wiener processes. In fact, $\tilde{w}_i(t) = w_i(\epsilon t)/\sqrt{\epsilon}$. Since $x_0^\epsilon(\cdot)$ is "nearly constant" over "long" time intervals for small ϵ, (5.4) suggests that, on the appropriate long time interval, we can treat $z_0^\epsilon(\cdot)$ as though $x_0^\epsilon(\cdot)$ were fixed, and use this fixed-x process to average out the the $z^\epsilon(\cdot)$. With this in mind, we define the *fixed-x process* $z_0(\cdot|x)$ (written simply as $z_0(\cdot)$ if the parameter is known) by

$$dz_0 = H(x, z_0) + v(x, z_0)d\tilde{w}_2. \tag{5.5}$$

Let A_x^ϵ denote the differential generator of the fixed-x process with parameter x, A^ϵ the differential generator of $(x^\epsilon(\cdot), z^\epsilon(\cdot))$, and $A^{0,\epsilon}$ that of $(x_0^\epsilon(\cdot), z_0^\epsilon(\cdot))$. We will use the following assumption.

A5.1. *For each initial condition and each x, (5.5) has a unique weak sense solution. For each x, $z_0(\cdot|x)$ has a unique invariant measure $\mu_x(\cdot)$. There is a continuous matrix $\bar{\sigma}(\cdot)$ such that*

$$\bar{\sigma}(x)\bar{\sigma}'(x) = \bar{a}(x) = \int \sigma(x,z)\sigma'(x,z)\mu_x(dz).$$

The factorization assumption on $\bar{a}(\cdot)$ is a convenience for the notation, since it allows a simple and concrete representation of the averaged process, but it is not necessary for the validity of the basic results.

The averaged system. Define the averaged functions and system by

$$\bar{G}(x, \alpha) = \int G(x,z,\alpha)\mu_x(dz), \quad \bar{k}(x,\alpha) = \int k(x,z,\alpha)\mu_x(dz),$$

$$dx = dt\left[\int \bar{G}(x,\alpha)m_t(d\alpha)\right] + \bar{\sigma}(x)dw \tag{5.6}$$

Here $m(\cdot)$ is an admissible relaxed control with respect to the standard vector-valued Wiener process $w(\cdot)$.

Let $\bar{A}^\alpha$ denote the differential generator of (5.6) with control fixed at α, and let $\bar{A}^m$ be that associated with the relaxed control $m(\cdot)$. We will use the following assumptions.

A5.2
$$G(x,z,\alpha) = G_0(x,z) + G_1(x,\alpha),$$

$$k(x,z,\alpha) = k_0(x,z) + k_1(x,\alpha).$$

The $\sigma(\cdot)$ and the $G_i(\cdot)$ are continuous and have at most a linear growth in x as $|x| \to \infty$, uniformly in α, z. The $k_i(\cdot)$ are bounded and continuous.

A5.3 *$H(\cdot)$ and $v(\cdot)$ are continuous and have at most a linear growth in z, uniformly in x.*

$$\bar{G}_0(x) = \int G_0(x,z)\mu_x(dz), \quad \bar{k}_0(x) = \int k_0(x,z)\mu_x(dz).$$

The cost function. Define the pathwise and mean costs:

$$\begin{aligned}
\gamma_T^\epsilon(x,z,m^\epsilon,w_1,w_2) &= \frac{1}{T}\int_0^T \int_U k(x^\epsilon(s),z^\epsilon(s),\alpha)m_s^\epsilon(d\alpha)ds, \\
\gamma^\epsilon(x,z,m^\epsilon,w_1,w_2) &= \limsup_T E\gamma_T^\epsilon(x,z,m^\epsilon,w_1,w_2), \\
\gamma_0^\epsilon(x,z) &= \inf_{\text{adm. controls}} \gamma^\epsilon(x,z,m^\epsilon,w_1,w_2).
\end{aligned} \tag{5.7}$$

$$\begin{aligned}
\gamma_T(x,m,w) &= \frac{1}{T}\int_0^T \int_U \bar{k}(x(s),\alpha)m_s^\epsilon(d\alpha)ds, \\
\gamma(x,m,w) &= \limsup_T E\gamma_T(x,m,w), \\
\gamma_0(x) &= \inf_{\text{adm. controls}} \gamma(x,m,w).
\end{aligned} \tag{5.8}$$

We keep the arguments m^ϵ, w_1, w_2 in γ^ϵ since the value depends on the joint distribution of the triple m^ϵ, w_1, w_2 and not just on m^ϵ. If the initial condition x is omitted in $\gamma_0(x)$ or in $\gamma(x,m,w)$, then the values do not depend on x.

Note that neither $\gamma_T^\epsilon(\cdot)$ nor γ_T involve expectations. They are simply pathwise averages. The main results are Theorems 5.1 and 5.2 below. Loosely speaking, we will show that as $\epsilon \to 0$ and $T \to \infty$, the pathwise average cost $\gamma_T^\epsilon(x,z,m^\epsilon,w_1,w_2)$ will converge to the mean cost per unit time for an averaged problem, where the system is (5.6) and the cost is $\gamma(x,m,w)$. Also, under broad conditions, given $\delta > 0$ there will be some sequence $\{m^\epsilon(\cdot)\}$ such that $\gamma_T^\epsilon(x,z,m^\epsilon,w_1,w_2)$ converges in probability to within δ of the infimum γ_0 of the costs for the averaged problem, and there is no admissible $\{m^\epsilon(\cdot)\}$ for which the limit is less than γ_0. The way that $\epsilon \to 0$ and $T \to \infty$ will not be important. Such results help to justify the use of the averaged problem even on the infinite time interval. We will use the condition

A5.4. *For each $\delta > 0$ and $\epsilon > 0$, there is a δ-optimal (for the cost functional $\gamma^\epsilon(x,z,m^\epsilon,w_1,w_2)$) admissible $m^\epsilon(\cdot)$ such that the corresponding sequence*

$$\{x^\epsilon(t), z^\epsilon(t), t < \infty, \epsilon > 0\}$$

is tight.

The condition (A5.4) implies that there are δ-optimal policies for which the systems $(x^\epsilon(\cdot), z^\epsilon(\cdot))$ do not explode as $t \to \infty$ and $\epsilon \to 0$.

We follow the terminology of Section 4. Define the functional occupation measures $\tilde{P}^{\epsilon,t}(\cdot)$ and $\tilde{P}^{\epsilon}_T(\cdot)$ by

$$\tilde{P}^{\epsilon,t}(B_0, C_0) = \delta_{\{z^\epsilon_t(\cdot)\}}(B_0)\delta_{\{\Delta_t m^\epsilon(\cdot)\}}(C_0)$$

$$\tilde{P}^\epsilon_T(\cdot) = \frac{1}{T}\int_0^T \tilde{P}^{\epsilon,t}(\cdot)dt.$$

Theorem 5.1. *Assume (A5.1) to (A5.4), and use the relaxed controls of (A5.4) for some $\delta > 0$. Then the sequence of $\mathcal{P}(S_0)$-valued random variables $\{\tilde{P}^\epsilon_T(\cdot), \epsilon > 0, T < \infty\}$ is tight. Let $\tilde{P}(\cdot)$ denote a weak limit (for some sequence $\epsilon_n \to 0, T_n \to \infty$) with values $\tilde{P}^\omega(\cdot)$. For almost all ω, $\tilde{P}^\omega(\cdot)$ induces a stationary process $(\tilde{z}^\omega(\cdot), \tilde{m}^\omega(\cdot))$ on S_0 in the sense that the distribution of $(\tilde{z}^\omega_t(\cdot), \Delta_t\tilde{m}^\omega(\cdot))$ does not depend on t. For almost all ω, there is a standard vector-valued Wiener process $\tilde{w}^\omega(\cdot)$ such that $\tilde{m}^\omega(\cdot)$ is admissible with respect to $\tilde{w}^\omega(\cdot)$ and $(\tilde{z}^\omega(\cdot), \tilde{m}^\omega(\cdot), \tilde{w}^\omega(\cdot))$ satisfy (5.6)*

$$\gamma^{\epsilon_n}_{T_n}(x, z, m^{\epsilon_n}, w_1, w_2) \Rightarrow \gamma(\tilde{m}, \tilde{w})$$

$$\gamma(\tilde{m}^\omega, \tilde{w}^\omega) = \tilde{E}^\omega\int_U\int_0^1 \tilde{k}(\tilde{z}^\omega(s), \alpha)\tilde{m}^\omega_s(d\alpha)ds.$$

Remark. In the last equation $\gamma(\tilde{m}^\omega, \tilde{w}^\omega)$ is the cost for the stationary problem, where the initial condition $\tilde{z}^\omega(0)$ is the random variable with the stationary distribution.

Proof. The $\{z^\epsilon_t(\cdot), t < \infty, \epsilon > 0\}$ is tight due to (A5.2) and (A5.4). The set $\{\Delta_t m^\epsilon(\cdot), t < \infty, \epsilon > 0\}$ is always tight. These facts imply the asserted tightness of $\{\tilde{P}^\epsilon_T(\cdot), \epsilon > 0, T < \infty\}$, analogously to the case of Theorem 4.1. Let $F(\cdot), h(\cdot), t_i, \psi_j(\cdot), p, q, \tau, s$ be as in Theorem 4.1, and let $\tilde{E}^\omega_T$ denote the expectation with respect to $\tilde{P}^{\epsilon,\omega}_T(\cdot)$. Then, analogously to the calculation in Theorem 4.1, by Ito's Formula and the definition of $\tilde{P}^\epsilon_T(\cdot)$ we have

$$\int F(z, \nu)\tilde{P}^\epsilon_T(dz\, d\nu) \stackrel{D}{=}$$

$$\frac{1}{T}\int_0^T dt\Big(h(z^\epsilon_t(t_i), (\Delta_t m^\epsilon, \psi_j)_{t_i}, i \le q, j \le p)\Big[f(z^\epsilon_t(\tau + s))$$

$$-f(z^\epsilon_t(\tau)) - \int_\tau^{\tau+s} du\int_U f'_x(z^\epsilon_t(u))G(z^\epsilon_t(u), z^\epsilon_t(u), \alpha)(\Delta_t m^\epsilon)_u(d\alpha)$$

$$-\frac{1}{2}\int_\tau^{\tau+s} du \text{ trace } f_{xx}(z^\epsilon_t(u)) \cdot a(z^\epsilon_t(u), z^\epsilon_t(u))\Big]\Big)$$

$$= \frac{1}{T}\int_0^T dt \int_\tau^{\tau+s} du \Big(h(x_i^\epsilon(t_i),(\Delta_i m^\epsilon, \psi_j)_{t_i}, i \le q, j \le p) \cdot$$

$$\cdot f_x'(x_1^\epsilon(u))\sigma(x_1^\epsilon(u)), z_1^\epsilon(u))dw_1(u)\Big). \tag{5.8}$$

It can be shown that, by using the averaging method of Theorem 3.2 we can replace the $G(\cdot)$ and $a(\cdot)$ in (5.8) by $\bar{G}(\cdot)$ and $\bar{a}(\cdot)$ without changing the limits as $\epsilon \to 0$ and $T \to \infty$. Only a few of the details will be given. Consider the term (5.9) which is a component of the third line of (5.8):

$$\int_\tau^{\tau+s} f_x'(x_1^\epsilon(u))G_0(x_1^\epsilon(u), z_1^\epsilon(u))du \tag{5.9}$$

Let $s = n\delta$ for an integer n and $\delta > 0$, and rewrite the integral as

$$\sum_{i=0}^{n-1} \delta \frac{1}{\delta} \int_{\tau+i\delta}^{\tau+i\delta+\delta} f_x'(x_1^\epsilon(u))G_0(x_1^\epsilon(u), z_1^\epsilon(u))du. \tag{5.10}$$

Define the shifted rescaled processes $x_{10}^\epsilon(u) = x^\epsilon(t+\epsilon u)$, $z_{10}^\epsilon = z^\epsilon(t+\epsilon u)$. Now take one of the summands in (5.10) and change the time scale $s \to \epsilon s$ to get

$$\delta \frac{\epsilon}{\delta} \int_{(\tau+i\delta)/\epsilon}^{(\tau+i\delta+\delta)/\epsilon} f_x'(x_{10}^\epsilon(u))G_0(x_{10}^\epsilon(u), z_{10}^\epsilon(u))du. \tag{5.11}$$

We study (5.11) as $\delta \to 0$ and $\epsilon/\delta \to 0$. Note that for each $\delta_0 > 0$,

$$\sup_{t<\infty} P\{\sup_{s\le\delta} |x_1^\epsilon(u) - x_1^\epsilon(0)| > \delta_0\} \to 0 \tag{5.12}$$

as $\delta \to 0$.

Define the occupation measures $\hat{P}^{\epsilon,\delta,s}(\cdot)$ and $\hat{P}_T^{\epsilon,\delta}(\cdot)$ by

$$\hat{P}^{\epsilon,\delta,s}(A \times B) = \delta_{\{x_{1+\tau+i\delta}^\epsilon(\epsilon s+\cdot)\}}(A)\delta_{\{z_{1+\tau+i\delta}^\epsilon(\epsilon s+\cdot)\}}(B),$$

$$\hat{P}_T^{\epsilon,\delta}(\cdot) = \frac{1}{T}\int_0^T \hat{P}^{\epsilon,\delta,s}(\cdot)ds.$$

Then (5.11) equals

$$\delta \int f_x(\phi(0))G_0(\phi(0),\xi(0))\hat{P}_{\delta/\epsilon}^{\epsilon,\delta}(d\phi d\xi).$$

By an "occupation measure argument" similar to that used with Theorem 3.2, and using the fact that $x_{1+\tau+i\delta}^\epsilon(\cdot)$ varies arbitrarily little over the interval $[0,\delta]$ (as implied by (5.12)), we can show that as $\delta/\epsilon \to 0$ and

$\epsilon \to 0$, the difference between $\hat{P}^{\epsilon,\delta}_{\delta/\epsilon}(\cdot)$ and the measure whose x-component is concentrated on $x^\epsilon_{i+\tau+i\delta}(\cdot)$ and whose z-component is concentrated on the stationary fixed-x $z_0(\cdot)$ process with parameter $x^\epsilon_{i+\tau+i\delta}(0)$ converges weakly to zero. This convergence is uniform in (t, τ). Thus

$$E\left|\frac{\epsilon}{\delta}\int_{(\tau+i\delta)/\epsilon}^{(\tau+i\delta+\delta)/\epsilon} f'_x(x^\epsilon_{i0}(u))G_0(x^\epsilon_{i0}(u), z^\epsilon_{i0}(u))du - \int f'_x(x^\epsilon_i(\tau+i\delta))G_0(x^\epsilon_i(\tau+i\delta), z)\mu_{x^\epsilon_i(\tau+i\delta)}(dz)\right| \to 0.$$

uniformly in τ, t. Thus we can replace the right hand part of the third line of (5.8) by

$$\int_\tau^{\tau+s}\int_U f'_x(x^\epsilon_i(u))\bar{G}(x^\epsilon_i(u), \alpha)\Delta_i m^\epsilon(d\alpha)du.$$

Similarly for the $\sigma(x, z)$ term in the fourth line of (5.8).

Analogously to the case of Theorem 3.2, the right hand side of (5.8) goes to zero in probability as $\epsilon \to 0$ and $T \to \infty$. We can also replace the $k(\cdot)$ in the cost function by $\bar{k}(\cdot)$ without changing the limits. Thus the cost functional has the same limits as those of

$$\int_{U}\left[\int_C \int \bar{k}(\phi(s), \alpha)\nu_t(d\alpha)\right]dt\bar{P}^\epsilon_T(d\phi d\nu).$$

The rest of the proof is no different than that used for Theorem 3.2 and the details are omitted. Q.E.D.

We define a *δ-optimal feedback control for the averaged system to be any measurable U-valued feedback control for which the averaged system (5.6) has a unique weak sense solution and a unique invariant measure and such that for any x, x' and admissible $(m(\cdot), u(\cdot))$, $\gamma(x, u^\delta) \leq \gamma(x', m, u) - \delta$. We now state the "approximate optimality theorem".*

Theorem 5.2. *Assume (A5.1)-(A5.9), and consider the averaged system (5.6) with cost function $\gamma(x, m, u)$. For each $\delta > 0$, let there be a continuous δ-optimal feedback control $u^\delta(\cdot)$ with the property that $\{x^\epsilon(t), z^\epsilon(t), \epsilon > 0, t < \infty\}$ is tight. Then*

$$\gamma^\epsilon_T(x, z, u^\delta, u_1, u_2) - \gamma(u^\delta) \leq \gamma_0 + \delta \tag{5.13}$$

in probability uniformly in each compact (x, z)-set, as ϵ and T go to their limits. For any $\delta_1 > 0$ and sequence of admissible $m^\epsilon(\cdot)$ satisfying (A5.4),

$$\limsup_{\epsilon, T} P\{\gamma^\epsilon_T(x, z, m^\epsilon, u_1, u_2) \leq \gamma_0 - \delta_1\} = 0 \tag{5.14}$$

Proof. (5.13) follows by Theorem 5.1, with the $m^\epsilon(\cdot)$ replaced by $u^\delta(\cdot)$. If (5.13) is not true, then there is a $\delta_0 > 0$ such that the limsup is greater than δ_0. Choose a weakly convergent subsequence of $\{\tilde{P}_T(\cdot)\}$ with the limit measure having the values $\tilde{P}^\omega(\cdot)$ and with $(\tilde{x}^\omega(\cdot), \tilde{m}^\omega(\cdot))$ denoting the induced process. Then

$$P\{\omega : \gamma(\tilde{m}^\omega, \tilde{u}^\omega) \leq \gamma_0 - \delta_1\} \geq \delta_0,$$

a contradiction to the δ-optimality of $u^\delta(\cdot)$ for small enough δ. Q.E.D.

6. REFLECTED DIFFUSIONS

We will do the analog of the method of Section 3 for a reflected diffusion. The "Skorohod problem" model of the reflected diffusion will be used [11]. The results of Sections 4 and 5 can readily be extended to this case. We will use the following assumption.

(A6.1) *Γ is the closure of a bounded open set in R^r with a twice continuously differentiable boundary $\partial\Gamma$. Let $n(x)$ denote the outward normal to $\partial\Gamma$ at x, and let $\beta(x)$ denote the reflection direction. Suppose that $\beta(x)$ is the restriction to $\partial\Gamma$ of a function which is twice continuously differentiable in a neighborhood of $\partial\Gamma$ and let there be $\alpha_0 > 0$ such that $-\beta'(x)n(x) \geq \alpha_0$, all $x \in \partial\Gamma$.*

The Skorohod problem. Let $w(\cdot)$ be a standard vector-valued $\mathcal{F}_t$ Wiener process. We say that $x(\cdot)$ solves the Skorohod problem if it is $\mathcal{F}_t$-adapted, continuous, and there is an $\mathcal{F}_t$-adapted function $Y(\cdot)$ such that for $x \in \Gamma$,

$$x(t) = x + \int_0^t b(x(s))ds + \int_0^t \sigma(x(s))dw(s) + Y(t),$$

$$|Y|(t) = \int_0^t I_{\{x(s)\}}(\partial\Gamma) d|Y|(s), \tag{6.1}$$

$$Y(t) = \int_0^t \beta(x(s)) d|Y|(s)$$

Define the shifted function $\Delta_t|Y|(\cdot) = |Y|(t + \cdot) - |Y|(t)$. Define the occupation measure $\hat{P}^t(\cdot)$ for the pair of processes $(x_t(\cdot), \Delta_t|Y|(\cdot))$, and then define $\tilde{P}_T(\cdot)$ as in (1.2). In order to get the needed tightness in the theorem below we will need the following result [11, Theorem 4.1]. Let $C^r[0, T]$ denote the space of continuous R^r-valued functions on the interval $[0, T]$ with the sup norm topology.

Theorem 6.1. *Assume (A6.1) and, for each $T < \infty$ consider the Skorohod problem*

$$x(t) = f(t) + k(t), \quad t \leq T_0. \tag{6.2}$$

where $f(\cdot)$ and $k(\cdot)$ are in $C^r[0,T]$ and

$$k(t) = \int_0^t \beta(x(s))d|k|(s)$$

$$|k|(t) = \int_0^t I_{\{x(s)\}}(\partial\Gamma)d|k|(s).$$

If $f(\cdot)$ is in a compact set, then $(x(\cdot), k(\cdot), |k|(\cdot))$ are in a compact set in $C^r[0,T]$.

Theorem 6.2. *Assume (A1.1) and (A6.1). Then $\{x_t(\cdot), \Delta_t|Y|(\cdot), t < \infty\}$ and $\{\tilde{P}_T(\cdot), T < \infty\}$ are tight. Let $\tilde{P}(\cdot)$ denote the limit of a weakly convergent subsequence of $\{\tilde{P}_T(\cdot), T < \infty\}$. Then, for almost all ω, the sample value $\tilde{P}^\omega(\cdot)$ induces a stationary process $\tilde{x}^\omega(\cdot)$ satisfying (6.1). Also (3.2) holds.*

Proof. The proof is similar to that of Theorem 3.2. The tightness of $\{x_t(\cdot), \delta_t|Y|(\cdot), t < \infty\}$ is a consequence of Theorem 6.1, and the tightness of the sequence of processes

$$\left\{\int_0 b(x_s(u))du, \int_0 \sigma(x_s(u))du_s(u), s < \infty\right\}$$

The tightness of the sequence of measure-valued random variables follows from the tightness of the above set of processes. Let $\tilde{P}(\cdot)$ denote the limit of a weakly convergent subsequence of $\{\tilde{P}_T(\cdot), T < \infty\}$, and let $(\tilde{x}^\omega(\cdot), |\tilde{Y}^\omega|(\cdot))$ denote the process induced by the value $\tilde{P}^\omega(\cdot)$. The $\tilde{x}^\omega(\cdot)$ and $|\tilde{Y}^\omega|(\cdot)$ are continuous processes and $x^\omega(t) \in \Gamma$ for all t. Also $|\tilde{Y}^\omega|(\cdot)$ can increase only when $\tilde{x}^\omega(t) \in \partial\Gamma$. The stationarity of the limit processes is proved as in Theorem 3.2.

We need only characterize the limits $\tilde{x}^\omega(\cdot)$. Let $\phi(\cdot)$ denote the generic path in $D^r[0,\infty)$ associated with the process $\tilde{x}^\omega(\cdot)$, and $y(\cdot)$ the generic path in $D^r[0,\infty)$ associated with the process $|\tilde{Y}^\omega|(\cdot)$. Redefine the function $F(\cdot)$ used below (3.3) as follows

$$F(\phi(\cdot), y(\cdot)) = h(\phi(t_i), y(t_i), i \le q)\Big[f(\phi(t+s)) - f(\phi(t)) -$$
$$\int_t^{t+s} Af(\phi(u))du - \int_t^{t+s} f_x'(\phi(u))\beta(\phi(u))dy(u)\Big].$$

This function is defined for all $\phi(\cdot) \in D^r[0,\infty)$ and for all $y(\cdot)$ in $D^r[0,\infty)$ which are of bounded variation. If $y(\cdot)$ is not of bounded variation, set the value of the function equal to some very large value. Define $\mathcal{F}_t^\omega = \mathcal{B}(\tilde{x}^\omega(s), |\tilde{Y}^\omega|(s), s \le t)$. Then

$$f(\tilde{x}^\omega(t)) - f(\tilde{x}^\omega(0)) - \int_0^t Af(\tilde{x}^\omega(u))du$$
$$- \int_0^t f_x'(\tilde{x}^\omega(u))\beta(\tilde{x}^\omega(u))d|\tilde{Y}^\omega|(u)$$

is an $\mathcal{F}_t$-martingale. This implies that there is an $\bar{u}^*(\cdot)$ such that the triple $(\bar{x}^*(\cdot), \bar{Y}^*(\cdot), \bar{u}^*(\cdot))$ satisfies the Skorohod problem (6.1). Q.E.D.

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